where the prime now denotes differentiation with respect to ξ. Equation (12) is a confluent hypergeometric equation having the following series solution:

$$g(\xi) = 1 - \frac{\lambda'}{\frac{1}{2}}\,\xi + \frac{\lambda'}{\frac{1}{2}} \cdot \frac{\lambda' + 1}{\frac{1}{2} + 1} \cdot \frac{\xi^2}{2!} - \frac{\lambda'}{\frac{1}{2}} \cdot \frac{\lambda' + 1}{\frac{1}{2} + 1} \cdot \frac{\lambda' + 2}{\frac{1}{2} + 2} \cdot \frac{\xi^3}{3!} + \ldots \quad (13)$$

Since $f'(\eta) = 2\eta g'(\xi)$, the boundary condition $f' = 0$ at $\eta = 0$ is satisfied by the requirement that g' be finite at the origin. The second complementary solution to Eq. (12) is not regular at the origin, and it is discarded. Equation (13) also can be expressed in a series (which terminates under certain conditions) as

$$g_n = e^{-\xi} \cdot \left[1 - \frac{n}{\frac{1}{2}}\,\xi + \frac{n(n-1)}{\frac{1}{2}(\frac{1}{2}+1)} \cdot \frac{\xi^2}{2!} - \frac{n(n-1)(n-2)}{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)} \cdot \frac{\xi^3}{3!} + \ldots\right] \quad (14)$$

where

$$n = \lambda' - \tfrac{1}{2} \quad (15)$$

Now it can be shown that, if n is zero or a positive integer, $g_n \rightarrow e^{-\xi} \cdot \xi^n$ as $\xi \rightarrow \infty$, thus satisfying the boundary condition $u_1 \rightarrow 0$ as $y \rightarrow \infty$. For negative values of n, $g_n \rightarrow \infty$ as $\xi \rightarrow \infty$, whereas for nonintegral positive values of n, $g \rightarrow \xi^{-\lambda'}$. Hence the only permissible functions are those corresponding to positive integral values of n. The solution given by Eq. (14) now may be written as

$$g_n = e^{-\eta^2} \cdot [H_{2n}(\eta)/H_{2n}(0)] \quad (16)$$

where $H_{2n}(\eta)$ are Hermite polynomials provided that $n = \lambda - (\sigma c_1/4\rho\nu) - \frac{1}{2}$ is a positive integer.

Since Eq. (4) is linear, a general solution can be written as

$$u_1 = \sum_{n=0}^{\infty} A_n \cdot \left(\frac{x}{x_0}\right)^{-[1/2 + \sigma c_1/4\rho\nu] - n} \cdot g_n(\xi) \quad (17)$$

which can be made to fit to some initial condition $u_{10}(y) = u_1(x_0, y)$ at the station $x = x_0$ where $C_0 = [4\nu x_0/U]^{1/2}$. Thus, u_{10} becomes a function of ξ, and Eq. (17) gives

$$u_{10}(\xi) = \sum_{n=0}^{\infty} A_n\, g_n(\xi) \quad (18)$$

It can be shown from Eq. (12) that the functions $g_n(\xi)$ are orthogonal in the interval 0 to ∞ with respect to the weighting function $\xi^{-1/2} \cdot e^{\xi}$. Thus the coefficients A_n may be written as

$$A_n = \frac{\int_0^\infty u_{10}(\xi) \cdot \xi^{-1/2} e^{\xi} g_n d\xi}{\int_0^\infty \xi^{-1/2} e^{\xi}\, g_n{}^2\, d\xi} \quad (19)$$

It is clear from (17) that the scale of the velocity defect inside the wake dies more quickly in the presence of a magnetic field than when the field is absent. Physically, this result is to be expected, as the effect of a transverse magnetic field is to make the velocity profile flatter and more uniform as in the case of Hartmann flow between two parallel plates. It also may be noted that the flux of momentum

$$\rho U \int_{-\infty}^{\infty} u dy$$

which is a measure of the resistance of the body is not independent of x but varies as $x^{-\sigma c_1/4\rho\nu}$ at large distance from the body, since $u_1 \sim x^{-[1/2 + \sigma c_1/4\rho\nu]} \cdot e^{-\eta^2}$ there. The invariance of the flux of momentum in the nonmagnetic case with respect to x is a consequence of the fact that the pressure is assumed constant inside the wake at a large distance from the body [although strictly speaking, inside the wake $p = P - (c_1 x/r^2) + 0(1/r^2)$, P being the freestream pressure], and this is not so in the present case due to the presence of the variable magnetic field, giving rise to a variable magnetic pressure. For a self-propelled body, $u_1 \sim x^{-[3/2 + (\sigma c_1/4\rho\nu)]} \cdot H_2(\eta)$, $H_2(\eta)$ being a Hermite polynomial.

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Local Solutions to the Two-Body Problem

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A nearly exact solution to the problem of two-bodies is given about a local point, in terms of the deviation in radial distance from the initial radius. When the deviation is small, as in the case of small eccentricity, the solution is competitive with that achieved by solving Kepler's equation. For higher eccentricities the solution holds with restraint on the time interval, without the need for specific attention to the crossover to parabolic and hyperbolic orbits. Two forms of the solution are manifested: one in terms of circular functions; the other in terms of hyperbolic functions.

Introduction

THE problem of two-bodies begins in effect with the attempt to solve a nonlinear differential equation. Many ingenious approaches have evolved; but, unfortunately, the character of the equation leads inevitably to a singular solution at the parabolic orbit. Indeed, the failure of unembellished solutions to Kepler's equation to provide the means for precision orbit determination has attracted the efforts of the most brilliant minds in mathematics and astronomy. Expansions about Barker's solution in parabolic orbit are legion.

In the following paper is an attempt to solve the two-body problem for radial distance in terms of the time by expanding r about r_0 and solving for the resultant Δr. This approach converts the original nonlinear equation to a linear equation when terms of degrees two and higher are neglected, yielding an excellent solution for small eccentricity orbits in terms of the circular functions; and it also yields excellent solutions for higher eccentricity orbits in terms of both the circular and hyperbolic functions, when smaller increments in Δr are taken.

It is of supreme importance to have a sound appreciation of the implications of this simple solution. Not only is it valid through the range of eccentricities, but it can easily be improved by solution of the equations containing the second- and third-degree terms, in terms of the Jacobi elliptic func-

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tions. It is planned to present solutions to the equations containing higher degree terms in future papers.

Derivation of the Solution

The radial equation of motion in two-body theory is given as

$$\ddot{r} + \mu/r^2 - \mu a(1 - e^2)/r^3 = 0$$

By proper selection of the system of units, one may set the gravitational constant μ and the major semiaxis a both equal to 1. In this system, the angle traversed in unit time over a circular orbit of unit radius is 1 rad. Then,

$$\ddot{r} + 1/r^2 - (1 - e^2)/r^3 = 0\dagger \qquad (1)$$

where e is the eccentricity. Let

$$r = r_0 + \Delta r \qquad \Delta r/r_0 = \alpha$$

Then,

$$1/r^2 = 1/r_0^2(1 - 2\alpha)$$

neglecting terms of higher degrees, and

$$1/r^3 = 1/r_0^3(1 - 3\alpha) \qquad \ddot{r} = \Delta\ddot{r} = r_0\ddot{\alpha}$$

Then, from Eq. (1),

$$\ddot{\alpha} + \frac{1 - 2\alpha}{r_0^3} - \frac{(1 - e^2)(1 - 3\alpha)}{r_0^4} = 0 \qquad (2)$$

Rearranging terms,

$$\ddot{\alpha} + \left[\frac{3(1 - e^2)}{r_0^4} - \frac{2}{r_0^3}\right]\alpha = \frac{(1 - e^2)}{r_0^4} - \frac{1}{r_0^3}$$

Let

$$\omega^2 = 3(1 - e^2)/r_0^4 - 2/r_0^3 \qquad (2.1a)$$

or

$$\omega^2 = 2/r_0^3 - 3(1 - e^2)/r_0^4 \qquad (2.1b)$$

so that $\omega^2 > 0$;

$$C = (1 - e^2)/r_0^4 - 1/r_0^3 = \tfrac{1}{3}(\pm\omega^2 - 1/r_0^3)$$

(Note that ω is positive, a frequency or growth-decay rate.) Then the solution to (2) is

$$\alpha = 2C/\omega^2 \sin^2\omega t/2 + \dot{r}_0/(\omega r_0) \sin\omega t \qquad (3a)$$

$$\alpha = -2C/\omega^2 \sinh^2\omega t/2 + \dot{r}_0/(\omega r_0) \sinh\omega t \qquad (3b)$$

When $\omega = 0$, Eqs. (3) reduce to

$$\alpha = (\dot{r}_0 t/r_0) + \tfrac{1}{6}t^2/r_0^3 + \ldots$$

in the circular function case, and reduce to

$$\alpha = (\dot{r}_0 t/r_0) - \tfrac{1}{6}t^2/r_0^3 + \ldots$$

by the use of hyperbolic functions. Of course, since Δr was assumed small, one could not expect agreement past the first-degree term in a rectilinear case.

In order for orbits to have solutions of type (3b), one finds that the condition (2.1b) implies $r_0 \geq \frac{3}{2}(1 - e^2)$. Inasmuch as r_0 attains a maximum value of $1 + e$,‡ one finds that $1 + e \geq \frac{3}{2}(1 - e^2)$, or $e \geq \frac{1}{3}$. That is, (3b) will not occur as a solution unless $e \geq \frac{1}{3}$.

A simple check of Eq. (3a) can be made for small e by taking periapsis as starting point. Here,

$$r_0 = 1 - e \qquad \dot{r}_0 = 0$$

$$\omega^2 = \frac{1 + 3e}{(1 - e)^3} \qquad C = \frac{e}{(1 - e)^3}$$

and one has

$$\alpha = \frac{2e}{1 + 3e} \sin^2\left[\frac{1 + 3e}{(1 - e)^3}\right]^{1/2} \frac{t}{2} \qquad (4)$$

One knows, for example, that at apapsis ($t = \pi$), α must equal $2e/(1 - e)$. Equation (4), therefore, yields a fair approximation even in this extreme case, being correct to the first degree in e.

The change in true anomaly v from initial value v_0 is given by the expression relating angular momentum magnitude h to dv/dt§:

$$dv/dt = h/r^2$$

Since

$$h = [\mu a(1 - e^2)]^{1/2} = (1 - e^2)^{1/2}$$

one has

$$\frac{dv}{dt} = \frac{(1 - e^2)^{1/2}}{r^2}$$

therefore

$$v - v_0 = (1 - e^2)^{1/2} \int_{t_0}^{t} \frac{dt^*}{r^2}$$

This integral is approximated as follows:

$$\frac{1}{r^2} = \frac{1}{r_0^2}(1 - 2\alpha)$$

$$v - v_0 = \frac{[1 - e^2]^{1/2}}{r_0^2} \int_{t_0}^{t} dt^*(1 - 2\alpha)$$

Using (3a) for α in the forementioned integral gives, for example,

$$v - v_0 = \frac{(1 - e^2)^{1/2}}{r_0^2}\left\{(t - t_0) - 2\left[\frac{C}{\omega^2}(t - t_0) - \frac{C}{\omega^3}(\sin\omega t - \sin\omega t_0) - \frac{\dot{r}_0}{\omega^2 r_0}(\cos\omega t - \cos\omega t_0)\right]\right\}$$

† In hyperbolic orbits, read $\ddot{r} + 1/r^2 - (e^2 - 1)/r^3 = 0$.

‡ Apapsis, elliptic orbits only.

§ Conventional methods may be more suitable (Kepler, Barker, Gauss).

Initial Evaluation of Perforated Ion Engine Emitters

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In the continuing engineering development aimed toward the improvement of lifetime and reliability in ion engines, the emitter remains one of the critical areas. Inasmuch as the prospective missions require that the emitter function for thousands of hours at high temperatures, an engineering program has been initiated to determine the feasibility and characteristics of emitters made from thin solid plates into

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